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An absorbing phase transition from a structured active particle phase

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Abstract

In this work we study the absorbing state phase transition of a recently introduced model for interacting particles with neighbourhood-dependent reproduction rates. The novelty of the transition is that as soon as the active phase is reached by increasing a control parameter a periodically arranged structure of particle clusters appears. A numerical study in one and two dimensions shows that the system falls into the directed percolation universality class.

1. Introduction

The study of continuous transitions into absorbing states has been of interest for decades [1–3]. Since the dynamics is trapped in the absorbing configuration and therefore it is irreversible, they constitute an interesting case of nonequilibrium phase transitions. Some examples showing such transitions include the contact process, epidemic spreading, directed percolation (DP) and reaction–diffusion systems. In general, the phase diagram of these systems involves two different phases: an absorbing one, in which the density field vanishes, and an active phase, where the expectation value of the density is different from zero. A high degree of universality has been found, and many of the known examples fall into the DP universality class [1–3].

Recently, in the context of population dynamics with possible applications to plankton populations or bacterial growth, a model where particles interact with other ones located within a finite distance was introduced [4]. The particles form clusters that are periodically arranged in the system for large values of a control parameter. When this parameter decreases a transition into an absorbing phase, with no particles *alive*, occurs. The novel feature is that the transition to the absorbing phase is directly performed from an active phase which shows a spatially periodic structure. According to the DP conjecture [5–7], all of the transitions into a unique absorbing phase, provided that interactions are short ranged, and that extra symmetries, memory effects and quenched disorder are absent, belong to the DP universality class. Since our model presents a transition characterized by the spatially periodic structure of the active phase,

which means that additional symmetries are broken, it is *a priori* possible that the system falls into a new, or at least, different universality class from DP.

In this paper we study the nature of this absorbing state phase transition in one and two dimensions. First we show the periodic spatial pattern that is formed in the system in the active phase growing from a localized seed. The structure function is then calculated showing the existence of a peak at nonzero wavenumber (signature of an structured phase) even for values of the control parameter very close to the critical one. The exponents characterizing the phase transition are then studied via spreading simulations from a particle seed. We will show that they are in complete agreement with the DP exponents, so that our system falls into the directed percolation universality class. We will argue that the insensitivity of the exponents of the absorbing transition to the periodic nature of the active phase may be due to the lack of a true long-range order of the periodic phase in the thermodynamic limit, so that we do not have actually an infinite number of active phases.

2. Model and spatial structures

The interacting particle model is introduced in [4] and further studied in [8–10]. It considers initially N_0 particles in the interval $[0, L]$ for the 1D case, and in the square $[0, L] \times [0, L]$ for 2D, with periodic boundary conditions. The particles are performing independent Brownian motions, leading to diffusion with diffusivity D. At every time step (time is discretized) a particle (say particle j) is chosen. It dies, disappearing from the system, with rate β_j, or reproduces, i.e. replicates itself, with rate λ_j. In this last case the newborn is placed at the same location as the parent particle. The death rate is taken to be constant, β_0, and the birth rate, λ_j, depends on the number of particles N_R^j within a distance R from its position, i.e.

$$\lambda_j = \max(0, \lambda_0 - N_R^j/N_s), \tag{1}$$

where the maximum condition is to assure the positivity of the rate, and λ_0, and N_s are constants, identical for all the particles. As shown in previous works, the control parameter of the system is $\mu = \lambda_0 - \beta_0$. We normalize time so that $\lambda_0 + \beta_0 = 1$ and then giving the value of μ fixes both λ_0 and β_0. It is clear that such interacting particle systems, where birth rates decrease in spatial areas of high particle density, has direct application in biological population dynamics [4, 8, 10] as models of competition for resources. More general particle models developed in the same spirit can be found in [11].

For an appropriate election of the parameters R and D (essentially a small D value is enough as shown in [4]), which we fix in all our calculations to be $R = 0.1$ and $D = 10^{-5}$, periodic spatial structures emerge in the system when we are in the active phase above a critical point μ_c. If $\mu < \mu_c$ particles become extinct. In figure 1 we show how these structures emerge from a localized initial pocket of particles. The plot is for one (left) and two (right) dimensions and $\mu = 0.7$ (the critical values of μ are later on computed to be $\mu_c^{1D} = 0.341(2)$ and $\mu_c^{2D} = 0.308(1)$ in one and two dimensions, respectively). In the one-dimensional case we plot the x coordinate of the particles in the horizontal axis, and the time variable on the vertical one. One can see that the number of particles increases and they begin to organize in periodically arranged clusters. At longer times the particle number stabilizes and a fluctuating periodic arrangement of particle clusters is established as the asymptotic state.

The system goes continuously, by increasing the value of the control parameter μ, from the absorbing configuration (no particles) to the active phase as can be seen in figure 2. There we plot the average stationary density, ρ_{st}, as a function of μ for one-dimensional systems with different sizes. This figure suggests a transition, when the system size goes to infinity, at around $\mu = 0.342$ (in good agreement with the more accurate values obtained below from seed

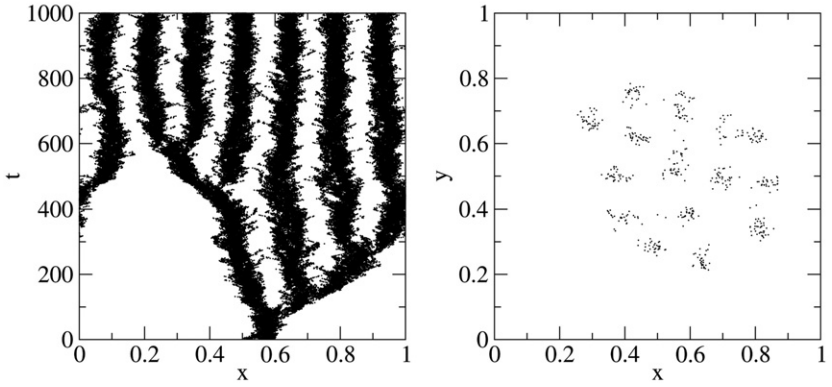

Figure 1. Left: 1D system. The spatial positions of the particles are on the x axis and time is on the y axis. A periodic array of fluctuating clusters is reached at long times. Right: 2D system. Spatial distribution of particles at time $t = 200$. The region with clusters expands and at long times the whole system is covered with a periodic arrangement (with hexagonal symmetry) of fluctuating clusters. In the two plots $\mu = 0.7$ (deep into the active phase), $D = 10^{-5}$, $N_s = 50$ and $R = 0.1$. Both structures emerge from a seed of 500 particles localized in the centre of the system.

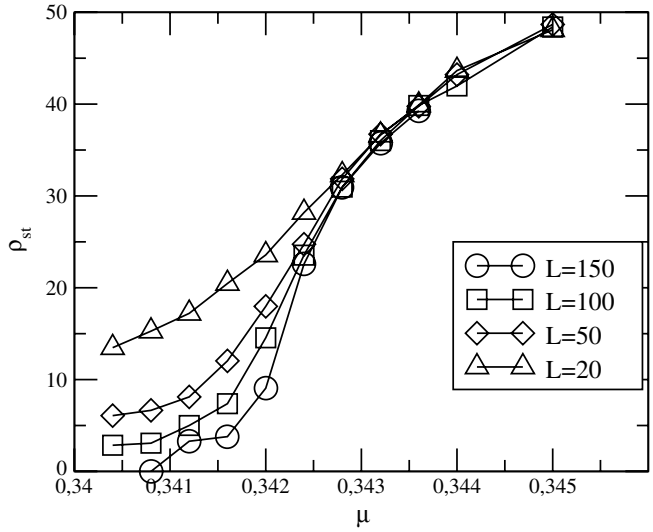

Figure 2. Average stationary density of particles for values of μ close to the critical value in the 1D system. The different system sizes considered are detailed in the legend box. Other parameters as in figure 1.

numerical experiments). The steady density is computed averaging over time in the steady state and also over many different realizations the number of particles and dividing by system size. The average over the realizations considers only those that do not enter into the absorbing phase. In two-dimensional systems a finite-size analysis of long-time steady states is computationally expensive and will not be attempted here. However, our following calculations of the exponents by seed spreading methods seem to confirm that we have also a continuum phase transition in two dimensions.

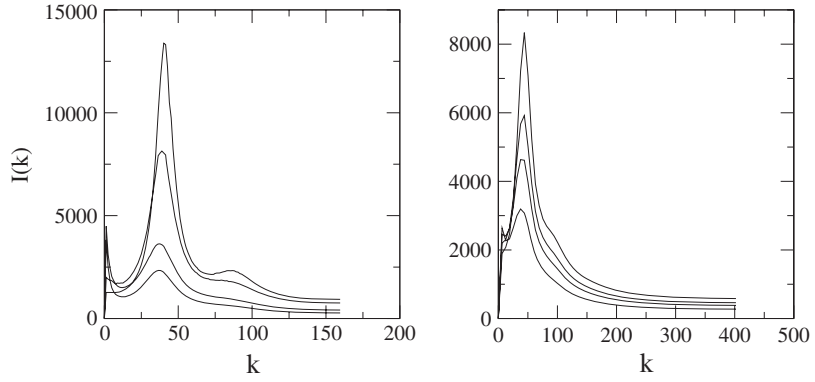

Figure 3. Structure factors for different values of the control parameter. For clarity, the large peak at $k = 0$ has been suppressed (i.e. we really plot $I(k) - I(0)\delta_{k,0}$). The left panel is for the 1D system with size $L = 5$ and, from top to bottom, $\mu = 0.38, 0.36, 0.3436, 0.3416$. The right panel corresponds to the 2D system so that the structure factor is circularly averaged. From top to bottom $\mu = 0.36, 0.34, 0.33, 0.32$. The rest of the parameters for both panels are the same as in figure 1.

A proper measure of the periodic spatial structure is given by the structure factor, $I(k) \equiv |\sum_{j=1}^{N(t)} e^{i\mathbf{k}\cdot\mathbf{x}_j}|^2$, where $N(t)$ is the number of particles in the system at time t and $\{\mathbf{x}_j\}$ are their positions. From the numerical computation of $I(k)$ we check whether the spatially structured character of the active phase is maintained as we approach from above the critical point, μ_c. In figure 3 we plot the structure factor in the one-dimensional case (left) for different values of μ (close to μ_c). The same quantity, but circularly averaged for the two-dimensional case, is shown to the right. In the 1D case the values of the control parameter are $\mu = 0.3416, 0.3436, 0.36, 0.38$, and in 2D, $\mu = 0.32, 0.33, 0.34, 0.36$. The largest peak of $I(k)$ is obtained at $k = 0$, and provides the square of total number of particles in the system. In the plots, in order to concentrate on the secondary peak which gives information about the spatial structure [12], the value of $I(0)$ is set to zero. On the other side, it is important to realize that the position of the secondary peak in $k = k_M \neq 0$ only slightly changes by increasing μ, which shows that the pattern structure is reached just when the active phase is developed, i.e. just above μ_c. Summing up, the fact that the secondary peak is different from zero and that the value of k_M remains almost invariant when different values of μ are considered, support the fact that the system is spatially structured even very close to the absorbing critical point, for both one and two dimensions. Moreover, the location of the peak gives information about the periodic structure by signalling the number of clusters n in the system (through $k_M = 2\pi n/L$) [8, 10].

3. Critical behaviour

By now we have shown that, at least in 1D, the transition from absorbing to active phase is continuous. Moreover the analysis with the structure factors has revealed that this transition occurs simultaneously with the appearance of a spatial pattern, i.e., the active phase always present a periodic structure. We now proceed to study the critical properties of this peculiar transition. The first step is to localize with a good confidence the critical value of the control parameter, μ_c, in both one and two dimensions. This is done by evolving in time a small seed of particles and monitoring the total number of particles, $N(t)$, averaged over all the runs [13, 14]. At the critical point it must scale asymptotically as a power law, $N(t) \propto t^{\eta}$, showing some curvature at sub- and super-critical values, which helps us to identify the critical point. This is shown in figure 4. We obtain $\mu_c = 0.341(2)$ in 1D, and $\mu_c = 0.308(1)$ in 2D.

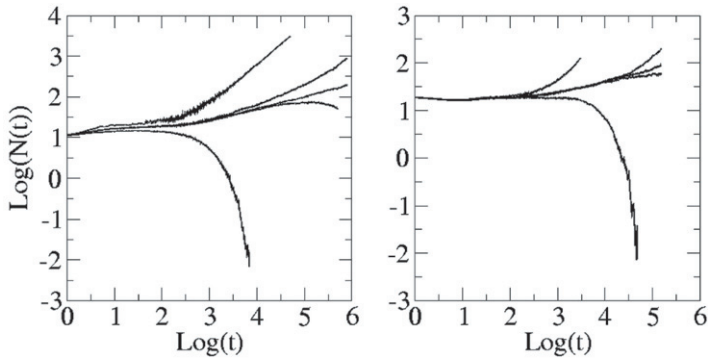

Figure 4. Time evolution of the total number of particles in the system. Left is for 1D and, from top to bottom, $\mu = 0.3440, 0.3424, 0.3412, 0.3404, 0.30$. Right is for 2D and, from top to bottom, $\mu = 0.32, 0.3084, 0.3081, 0.3080, 0.3040$. Other parameters as in figure 1. In both plots we average a number of realizations between 1200 and 2000. The critical value of μ, μ_c, is the one corresponding to the line in which upwards or downwards curvature is absent.

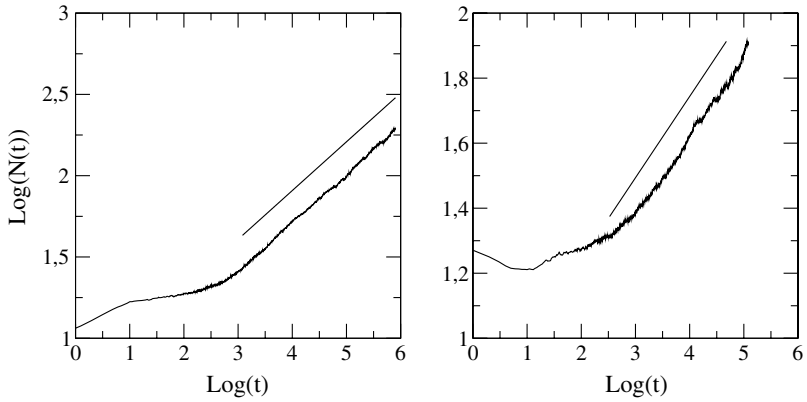

Figure 5. Number of particles versus time right at the critical point $\mu = \mu_c$ (averaged between 1200 and 2000 runs). Left is for 1D and right for 2D. Other parameters as in figure 1. The lines give the best fit to the plots, with slopes 0.30 (1D) and 0.25 (2D).

Besides $N(t)$, the total number of particles, we compute the survival probability, $P_s(t)$, which is the probability that the system remains in the active phase at time t, and the mean square radius, $R^2(t)$, defined as the mean square distance of all the particles with respect to the centre of mass of the system (remember that as the initial seed of particles grow many clusters form), and also averaged over all the runs. At the critical point there is no characteristic timescale and these magnitudes scale asymptotically as

$$\begin{aligned} P_s(t) &\propto t^{-\delta}, \\ R^2(t) &\propto t^{z}. \end{aligned} \tag{2}$$

In figures 5, 6 and 7 we plot, respectively, $N(t)$, P_s and R^2 versus time at the critical point for one and two dimensions. The slopes of the linear fits shown are, in 1D: $\eta = 0.30$, $\delta = 0.17$ and $z = 1.09$; in 2D: $\eta = 0.250$, $\delta = 0.43$ and $z = 0.72$. In all the plots the system size is chosen to be very large, $L = 100$, in order to avoid finite-size effects, and the total number of runs is between 1200 and 2000. These values are in good agreement with the DP critical exponents [1–3] so that one can conclude that the model studied is in the DP universality class.

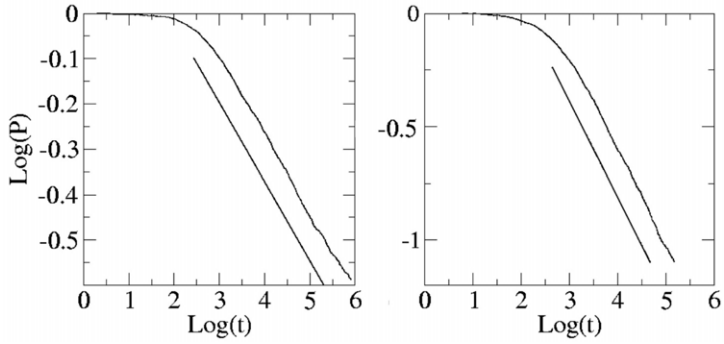

Figure 6. Survival probability versus time right at the critical point $\mu = \mu_c$. Left is for 1D and right for 2D. Other parameters as in figure 1. The lines give the best fit to the plots, with slopes -0.17 (1D) and -0.43 (2D).

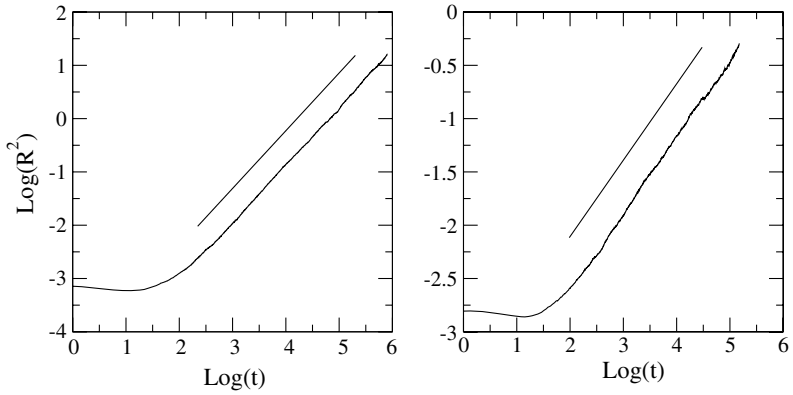

Figure 7. Mean square radius versus time right at the critical point $\mu = \mu_c$. Left is for 1D and right for 2D. Other parameters as in figure 1. The lines give the best fit to the plots, with slopes 1.09 (1D) and 0.72 (2D).

4. Discussion and summary

Despite the nonstandard features of the active phase in the absorbing phase transition considered here, our numerical results identify it as pertaining to the DP universality class. There is a naive analytical argument supporting this finding: an approximate Langevin equation for a stochastic quantity $\Phi(x, t)$ that under averaging gives the expected value of the local particle density was derived in [4] by using Fock space techniques. It has the form

$$\partial_t \Phi(\mathbf{x}, t) = D\nabla^2 \Phi(\mathbf{x}, t) + \mu\Phi(\mathbf{x}, t) - \frac{1}{N_s}\Phi(\mathbf{x}, t)\int_{|\mathbf{x}-\mathbf{r}|<R} \mathrm{d}\mathbf{r}\, \Phi(\mathbf{r}, t) + \eta(\mathbf{x}, t), \tag{3}$$

with $\eta(\mathbf{x}, t)$ being a noise term with zero mean and very complex correlations (see [4]), which depend on $\Phi(x, t)$ itself, so that the noise is multiplicative, and including terms of both types, spatially local and nonlocal. The important point, however, is that these correlations have a finite range (of order R). Although nonlocal, the interactions in equation (3) are also of finite range R. Therefore if we scale all lengths by the system size L and take the thermodynamic limit $L \to \infty$ to study critical properties, the relative interaction range R/L tends to zero and, after appropriate scaling of the interaction parameters, we recover a local partial differential

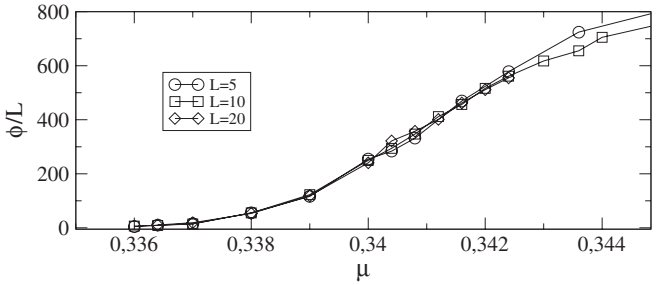

Figure 8. Ordering parameter ϕ (see the definition in the text) divided by the system size L versus the control parameter μ in the 1D case. The different values of L used in the plot are written down in the legend. Other parameters as in figure 1. The good collapse of the data is consistent with the absence of long-range periodic order.

equation that is nothing but the Reggeon field theory, i.e., the continuum description of the DP universality class. The idea is that nonlocal but finite-range interaction becomes local under the renormalization group microscope. This argument will fail, and corrections to the DP exponents may appear, if there are additional conserved quantities coupled to the dynamics. Since the deterministic part of equation (3) has periodic solutions [4, 8, 10], one such conserved quantity could be the phase of the periodic pattern, if the noise term in (3) turns out to be unable to restore the translational symmetry broken by the deterministic pattern forming process. Thus, a possible explanation of the persistence of DP exponents here is that the active phase does not really break translational symmetry in the thermodynamic limit, and thus we do not have an infinity of active phases but a single one with only short-range order.

A preliminary check of these ideas has been performed in one dimension (in nonequilibrium models, symmetry breaking can occur even in low dimensions [12]). To analyse the periodic ordering we define an order parameter ϕ as the value of the secondary peak of the structure factor (also averaged over time and realizations), i.e., $\phi = \langle I(k_M)\rangle$. Note that if particles are distributed at random then $\langle I(k \neq 0)\rangle = \langle N\rangle$ and typically $\langle N\rangle \propto L^d$, whereas for particles arranged with long-range periodic order at wavenumber k_M we have $\langle I(k_M)\rangle \propto L^{2D}$. Thus, the scaling with system size of the order parameter ϕ contains the needed information about the presence or absence of long-range order and thus of translational symmetry breaking.

In figure 8 we show ϕ/L versus μ for different system sizes in the 1D case. The good collapse of the data for the different sizes is consistent with the interpretation that long-range periodic order is suppressed in the thermodynamic limit (large L) at least in one dimension. If this preliminary result is maintained for larger sizes and for two-dimensional systems, then the DP exponents found here can be understood as a consequence of having short-range interactions and a single active phase, not an infinity of them. Further numerical work along these lines will be the subject of future research.

In summary, we have shown that a system of Brownian particles competing for the resources in their neighbourhood presents a continuous absorbing phase transition. The active phase is made of particle clusters periodically arranged but, despite this peculiarity, the absorbing transition falls into the DP universality class in both one and two dimensions.

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References

[1] Marro J and Dickman R 1999 *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge: Cambridge University Press)
[2] Hinrichsen H 2000 *Adv. Phys.* **49** 815–958
[3] Ódor G 2004 *Rev. Mod. Phys.* **76** 663–724
[4] Hernández-García E and López C 2004 *Phys. Rev.* E **70** 016216
[5] Janssen H K 1981 *Z. Phys.* B **42** 151–4
[6] Grassberger P 1982 *Z. Phys.* B **47** 365–74
[7] Jensen I 1993 *Phys. Rev.* E **47** R1–4
[8] López C and Hernández-García E 2004 *Physica* D **199** 223–34
[9] López C and Hernández-García E 2005 *Physica* A **356** 95–9
[10] Hernández-García E and López C 2005 *J. Phys.: Condens. Matter* **17** S4263–74
[11] Birch D A and Young W R 2006 *Theor. Popul. Biol.* **70** 26–42
[12] Bassler K E and Racz Z 1995 *Phys. Rev.* E **52** R9–12
[13] Muñoz M A, Grinstein G, Dickman R and Livi R 1996 *Phys. Rev. Lett.* **76** 451–4
[14] Muñoz M A, Grinstein G and Dickman R 1998 *J. Stat. Phys.* **91** 541–69